Exercise 3

Verify the Cauchy–Schwarz inequality and the triangle inequality for the vectors in Exercises 3 to 6.

$$\mathbf{x} = (2, 0, -1),\ \mathbf{y} = (4, 0, -2)$$

Solution

Cauchy–Schwarz Inequality

Check the Cauchy–Schwarz inequality $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ for the given vectors.

$$|\mathbf{x} \cdot \mathbf{y}| = |(2)(4) + (0)(0) + (-1)(-2)| = |10| = 10$$

$$\|\mathbf{x}\| = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{5}$$

$$\|\mathbf{y}\| = \sqrt{4^2 + 0^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$$

As a result,

$$|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\| = 10,$$

which means the Cauchy–Schwarz inequality is satisfied.

Triangle Inequality

Now check the triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for the given vectors.

$$\mathbf{x} + \mathbf{y} = (2, 0, -1) + (4, 0, -2) = (6, 0, -3)$$

$$\|\mathbf{x} + \mathbf{y}\| = \sqrt{6^2 + 0^2 + (-3)^2} = \sqrt{45} = 3\sqrt{5}$$

$$\|\mathbf{x}\| = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{5}$$

$$\|\mathbf{y}\| = \sqrt{4^2 + 0^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$$

As a result,

$$\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\| = 3\sqrt{5},$$

which means the triangle inequality is satisfied.